MAT 0100104 : CLASSICAL ALGEBRA

- Exponential and Logarithmic function of complex numbers.
- Hyperbolic functions.
- Algebraic equations : Deduction from Fundamental Theorem of classical Algebra
- Descartes' rule of signs.

Book : Higher Algebra by

S K Mappa

§ Exponential function :

When x is real, the infinite series

$1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$

converges for all x and the sum is denoted by e^x.

e^x is a function of a real variable x defined for all x. This is called the exponential function of a real variable x.

For a complex variable $z = x + iy$, the exponential function of z, written as $\exp z$, is defined by —

$$\exp(x+iy) = e^x(\cos y + i \sin y)$$

→ When z is purely real, $y = 0$,

and $\exp z = e^x(\cos 0 + i \sin 0)$

$\Rightarrow \exp x = e^x$

when z is purely imaginary, $x = 0$
and $\exp z = (\cos y + i \sin y)$

$$\Rightarrow \exp(iy) = \cos y + i \sin y$$

→ Since $e^x > 0$, $\forall$ real x,
$e^x(\cos y + i \sin y)$ represents a complex number in polar form, e^x being the modulus and y being an amplitude of $\exp z$.

Since $e^x \neq 0$ for any real number x, $\exp z$ is a non-zero complex number for any complex number z.

Let $u + iv$ be a non-zero complex number and let its polar representation be $r(\cos\theta + i \sin\theta)$.
Since r is positive, $\log r$ is real and r can be expressed as $r = e^{\log r}$

$$\therefore u + iv = e^{\log r}(\cos\theta + i \sin\theta)$$
$$= \exp(\log r + i\theta)$$

Thus, when $u + iv$ is a given non-zero complex number, there exists a complex number $z = \log r + i\theta$ such that $\exp z = u + iv$. Which means that the range of the exponential function of z is the entire complex plane excluding the origin.

Properties:

1. $\exp z_1 \cdot \exp z_2 = \exp(z_1 + z_2)$, where z_1, z_2 are complex numbers

Proof: Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$
Then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Thus for a non-zero complex number ...
there exist ...

$\exp z_1 = e^{x_1}(\cos y_1 + i\sin y_1)$

$\exp z_2 = e^{x_2}(\cos y_2 + i\sin y_2)$

$\exp z_1 . \exp z_2 = e^{x_1}(\cos y_1 + i\sin y_1) . e^{x_2}(\cos y_2 + i\sin y_2)$

$= e^{x_1+x_2}[\cos y_1 \cos y_2 + i\sin y_2 \cos y_1 + i\sin y_1 \cos y_2 + i^2 \sin y_1 \sin y_2]$

$= e^{x_1+x_2}[\cos y_1 \cos y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2) - \sin y_1 \sin y_2]$

$= e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)]$

$= \exp[(x_1+x_2) + i(y_1+y_2)]$

$= \exp(z_1 + z_2)$

proved.

2. $\frac{\exp z_1}{\exp z_2} = \exp(z_1 - z_2)$

Proof: Since $\exp z_2$ is a non-zero complex number, $\frac{\exp z_1}{\exp z_2}$ is defined.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$.

$\exp z_1 = e^{x_1}(\cos y_1 + i\sin y_1)$

$\exp z_2 = e^{x_2}(\cos y_2 + i\sin y_2)$

Now, $\frac{\exp z_1}{\exp z_2} = e^{x_1 - x_2} \frac{\cos y_1 + i\sin y_1}{\cos y_2 + i\sin y_2}$

$= e^{x_1-x_2} \frac{(\cos y_1 + i\sin y_1).(\cos y_2 - i\sin y_2)}{(\cos y_2 + i\sin y_2)(\cos y_2 - i\sin y_2)}$

$= e^{x_1-x_2} . \frac{\cos y_1 \cos y_2 - i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2 - i^2 \sin y_1 \sin y_2}{\cos^2 y_2 - i^2 \sin^2 y_2}$

NI... for a non-zero complex number w | > t... = n

$$= e^{x_1-x_2} \cdot \frac{\cos y_1 \cos y_2 - i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2 + \sin y_1 \sin y_2}{\cos^2 y_2 + \sin^2 y_2}$$

$$= e^{x_1-x_2}\left\{(\cos y_1 \cos y_2 + \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 - \cos y_1 \sin y_2)\right\}$$

$$= e^{x_1-x_2} \cdot \left(\cos(y_1-y_2) + i\sin(y_1-y_2)\right)$$

$$= \exp[(x_1-x_2) + i(y_1-y_2)]$$

$$= \exp(z_1-z_2)$$ proved.

COROLLARY: $\frac{1}{\exp z} = \exp(-z)$ $[\because \exp(0)=1]$

3. If n be an integer, $(\exp z)^n = \exp(nz)$

This is followed from the property 1 and the relation $(\exp z)^{-1} = \exp(-z)$.

4. If n be an fraction, say p/q, $(\exp z)^n$ has q distinct values but $(\exp(nz))$ is unique. $\exp(nz)$ is one of the values of $(\exp z)^n$.

5. If n be an integer, $\exp(z+2n\pi i) = \exp$
This follows from the property 1 and from the relation $\exp(2n\pi i) = 1$, which states that the $\exp z$ is a periodic function of z with period $2\pi i$.

Note: A function $f(z)$ is said to be a periodic function on its domain D if there exists a non-zero constant w s.t for all integers n, $f(z+nw) = f(z)$ — (1) holds for all $z \in D$. If no submultiple of w satisfies the relation (1), then w is said to be the period of $f(z)$.

6. Let w be a non-zero complex number. Then there always exists a complex no z such that $\exp z = w$.
By 5, $\exp z = w \Rightarrow \exp(z + 2n\pi i) = w$, $n \in \mathbb{Z}$

NI... for a non-zero complex number w | ... $= n$

$$= e^{x_1-x_2} \cdot \frac{\cos y_1 \cos y_2 - i \cos y_1 \sin y_2 + i \sin y_1 \cos y_2 + \sin y_1 \sin y_2}{\cos^2 y_2 + \sin^2 y_2}$$

$$= e^{x_1-x_2} \left\{ (\cos y_1 \cos y_2 + \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 - \cos y_1 \sin y_2) \right\}$$

$$= e^{x_1-x_2} \cdot \left(\cos(y_1-y_2) + i \sin(y_1-y_2) \right)$$

$$= \exp[(x_1-x_2) + i(y_1-y_2)]$$

$$= \exp(z_1 - z_2) \text{ proved.}$$

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Note: A function $f(z)$ is said to be a periodic function on its domain D if there exists a non-zero constant w s.t for all integers n, $f(z + nw) = f(z)$ —(1) holds for all $z \in D$. If no submultiple of w satisfies the relation (1), then w is said to be the period of $f(z)$.

6. Let w be a non-zero complex number Then there always exists a complex no. z such that $\exp z = w$.

By 5, $\exp z = w \Rightarrow \exp(z + 2n\pi i) = w$, $n \in \mathbb{Z}$

Thus for a non-zero complex number w there exist infinitely many complex numbers z such that $\exp z = w$.

ex1. Find all complex numbers z such that $\exp z = -1$.

Let $z = x + iy$

Then $\exp z = -1$

$\Rightarrow e^x(\cos y + i\sin y) = -1 + i.0$

$\Rightarrow e^x \cos y + i e^x \sin y = -1 + i.0$

$\Rightarrow \underbrace{e^x \cos y = -1}_{(1)}$ and $e^x \sin y = 0$ — (2)

Now, $(1)^2 + (2)^2 \Rightarrow (e^x \cos y)^2 + (e^x \sin y)^2 = (-1)^2 + 0^2$

$\Rightarrow e^{2x}(\cos^2 y + \sin^2 y) = 1$

$\Rightarrow e^{2x} = 1 = e^0$

$\Rightarrow 2x = 0$

$\Rightarrow x = 0$

Again,

$\frac{(2)}{(1)} \Rightarrow \frac{e^x \sin y}{e^x \cos y} = \frac{0}{-1}$

$\Rightarrow \tan y = 0$

$\Rightarrow \tan y = \tan 0$

$\Rightarrow y = (2n+1)\pi$, $n \in \mathbb{Z}$ (set of integers)

$\therefore z = x + iy$

$= 0 + i(2n+1)\pi$

$= (2n+1)\pi i$. //

ex2. Find all values of z such that $e^z = 1 + i$.

Sol^n: Let $z = x + iy$

$\Rightarrow \exp z = 1 + i$

$\Rightarrow e^{x+iy} = 1 + i$

$\Rightarrow e^x(\cos y + i\sin y) = 1 + i$

$\Rightarrow e^x \cos y + i e^x \sin y = 1 + i$

$\Rightarrow \underbrace{e^x \cos y = 1}_{(1)}$, $e^x \sin y = 1$ — (2)

Now, $(1)^2 + (2)^2$, $(e^x \cos y)^2 + (e^x \sin y)^2 = 1^2 + 1^2$

$\Rightarrow e^{2x} = 2$